Nonlinear nonlocal diffusion of magnetic flux in thin type-II superconductors and Josephson junction arrays: Exact solutions

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An exact solution of the nonlinear nonlocal diffusion problem is obtained that describes the evolution of the magnetic flux injected into a soft or hard type-II superconductor film or a two-dimensional Josephson junction array. (The magnetic field in vortices is assumed to be perpendicular to the film; the electric field induced by the vortex motion is proportional to the local magnetic induction; flux creep in the hard superconductors under consideration is described by the logarithmic $U(j)$ dependence.) Self-similar flux distributions with sharp square-root fronts are found. The fronts are shown to expand with power law time-dependence. A sharp peak in the middle of the distribution appears in the hard superconductor case.

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There are two kinds of problems of magnetic flux evolution in the type-II superconductors. In the first type problems the flux lines are parallel to the surfaces of the superconductor plate. In this case the motion of every vortex is determined by it's nearest environment, so local nonlinear diffusion takes place. It can be described by nonlinear differential equations[1−9], which are similar to classical equations for nonlinear diffusion (see, for example,[10−14]). In the second type problems the flux lines are perpendicular to the surface of a thin superconductor[15−18]. Then every vortex is influenced by *all* vortices in the superconductor, and one observes a so called *nonlocal* nonlinear diffusion of the vortices, which can described by a complicated integral differential equation. Note that the geometry of a flat sample in a perpendicular magnetic field is realized in most experiments which study the space-time image of the magnetic flux in type-II superconductors (see[20] and references therein). Very similar to that is the problem of the magnetic flux penetration into two-dimensional Josephson junction arrays[19]. Macroscopic dynamical equations for the magnetic flux in the last case are the same as for type-II syperconductor films.

Up to now the last geometry was theoretically studied only in the particular case of the flux flow resistivity independent of the magnetic field. The situation where the electric field E depends linearly on the current density j[15,16] was considered numerically. Switching on of a constant magnetic field in the situation, in which the electric field depends exponentially on j and the flux flow resistivity is again independent of the local magnetic induction B, was treated exactly in paper[17]. (Note that, strictly speaking, one can introduce the quantity "flux flow resistivity" only if $E \propto j$. When we use this term for brevity, we mean that $E(B,j)$ can be written in the form $E(B,j) = \rho(B)\epsilon(j)$, where ρ and ϵ are some functions.)

In the present paper we shall exactly describe the evolution of the magnetic flux injected into the film of a type-II superconductor, in which the flux flow resistivity is proportional to the local magnetic induction (that is the most widespread situation[21]). The cases of other field dependencies of the flux flow resistivity will be discussed only briefly. Note that the flux is assumed to be injected into a middle of the film instead of penetration of the magnetic flux from boundaries which is studied usually.

Let us consider the evolution of the magnetic flux injected in the infinite thin film (the xy plane) of a type-II superconductor (the flux lines are perpendicular to the surface). We assume the problem to be homogeneous along y, so the local magnetic induction B depends only on the coordinate x and on time. The current flows along y. An applied magnetic field is absent. We write out immediately the equation for $B(x,t)$ in the most general form (afterwards we shall show how it can be obtained):

$$\frac{\partial B\left(x\right)}{\partial t} = D\frac{\partial}{\partial x}$$
$$\times\left\{\left|B\left(x\right)\right|^q\left|\int_{-\infty}^{\infty}du\frac{B\left(u\right)}{u-x}\right|^p sign\left[\int_{-\infty}^{\infty}du\frac{B\left(u\right)}{u-x}\right]\right\}. \quad (1)$$

Here we have introduced the dimensional coefficient D and the constants $p > 0, q \geq 0$. It is convenient for us to use Eq. (1) rather than the equation for the current density (see[15,16]) or than the equation for the electric field (see[17]).

Let us first show how Eq. (1) can be obtained for a soft type-II superconductor. The continuity equation for the vortex density n ($B = \Phi_0 n$, Φ_0 is a flux quantum) has the form:

$$\frac{\partial n}{\partial t} + div\mathbf{J}_v = 0. \quad (2)$$

In the simplest case, the vortex flow $\mathbf{J}_v$ which is proportional to the electric field, can be expressed in terms of the vortex density and the vortex velocity: $J_v = nv$ (v is the velocity of vortices; the sign of $n(x)$ is plus if the magnetic field in vortices in the point under consideration is parallel to the z axis and it is minus otherwise; we assume that vortices and antivortices annihilate instantaneously). One can write for a soft superconductor:

$\eta v = (\Phi_0/c)\, j$, where j is the current density, c is the velocity of light, and the viscosity η is related to the normal phase resistivity ρ_n by the equation $\Phi_0/\eta = \rho_n c^2/H_{c2}$ (H_{c2} is the upper critical magnetic field). Thus we have for a soft superconductor in the considered geometry:

$$\frac{\partial B}{\partial t} = -\frac{\Phi_0}{c\eta}\frac{\partial}{\partial x}\left(|B|\, j\right). \tag{3}$$

This equation is also available in the case of the flux lines parallel to the surface of a soft superconductor - see[6], where we discuss the reasons for the modulus of B to appear. However, in this situation $j = -(1/4\pi c)\partial B/\partial x$, so

$$\frac{\partial B}{\partial t} = -\frac{\Phi_0}{4\pi c\eta}\frac{\partial}{\partial x}\left(|B|\,\frac{\partial B}{\partial x}\right). \tag{4}$$

In a thin infinite superconductor film of the thickness d, $B(x,t)$ related to the current density $j(x,t)$ by Ampére's law, which reads

$$B\left(x\right) = \frac{2d}{c}\int_{-\infty}^{\infty} du \frac{j\left(u\right)}{u-x}, \tag{5}$$

i.e. the Gilbert transformation[22]. It may be inverted to give

$$j\left(x\right) = -\frac{c}{2\pi^2 d}\int_{-\infty}^{\infty} du \frac{B\left(u\right)}{u-x}. \tag{6}$$

Inserting the expression (6) for j into Eq. (3), one obtains Eq. (1) with $q = p = 1$ and $D = \Phi_0/(2\pi^2 d\eta)$. (The corresponding stationary problem was considered in[23].)

Frequently, in the flux creep regime, one can assume that the effective creep activation barrier grows logarithmically with decreasing current: $U(j) = U_0 \ln|j_c/j|$, where U_0 is the characteristic scale of the pinning energy barrier, j_c is the critical current density, so the vortex velocity $v = l_h\omega_h \left|j/j_c\right|^{U_0/kT} j/j_c$, where l_h and ω_h are the averaged hopping distance and the hopping frequency correspondingly (see[1] and references therein). Using again Eqs. (3) and (6), we obtain Eq. (1) with $q = 1, p = 1 + U_0/kT, D = l_h\omega_h(c/2\pi^2 dj_c)^p$. If before the injection of the flux, there is the homogeneous magnetic induction B_0 in a sample, then for the extra magnetic induction $\delta B(x,t) = B(x,t) - B_0 \ll B_0$ one obtains Eq. (1) with $q = 0$, so real linearization is absolutely impossible for hard superconductors.

Let the magnetic flux $\Phi = \int dx B(x) > 0$ be injected in the sample at the initial moment. (We assume that $B(x)$ is nonzero in a restricted range around the point $x = 0$.) At long times one may search the solution in the following self-similar form:

$$B(x,t) = C_1 t^{-\beta} b\left(\frac{x}{C_2 t^{\alpha}}\right), \tag{7}$$

where C_1 and C_2 are constants. Inserting this form into Eq. (1) and comparing powers of t in all terms of the equation, we get the following relation for the exponents α and β:

$$\alpha + \beta(q + p - 1) = 1. \tag{8}$$

Using the condition of the flux conservation $\Phi = \int dx B(x,t) = const$, we get $\alpha = \beta = 1/(q+p)$. Thus solution to Eq. (1) can be written in the following self-similar form

$$B\left(x,t\right) = \left(\frac{tD}{\Phi}\right)^{-1/(q+p)} b\left(\frac{x}{\left(\Phi^{q+p-1} tD\right)^{1/(q+p)}}\right), \tag{9}$$

where $b(\xi)$ is defined by the equation

$$-\frac{1}{q+p}\frac{d}{d\xi}\left[\xi b(\xi)\right] = \frac{d}{d\xi}$$
$$\times\left\{|b(\xi)|^q \left|\int_{-\infty}^{\infty} d\zeta \frac{b(\zeta)}{\zeta - \xi}\right|^p sign\left(\int_{-\infty}^{\infty} d\zeta \frac{b(\zeta)}{\zeta-\xi}\right)\right\}. \tag{10}$$

Note that a full derivative appears in the right hand part of Eq. (10) because of the equality of the exponents α and β in the regime of conservation of the injected flux. Thus, for the values of ξ where $b(\xi)$ is nonzero, we get the equation

$$\int_{-\infty}^{\infty} d\zeta \frac{b(\zeta)}{\zeta-\xi} = -\left(\frac{1}{q+p}\right)^{1/p} b^{(1-q)/p}\left(\xi\right)\left|\xi\right|^{1/p} sign(\xi). \tag{11}$$

(We assume that $b(-\xi) = b(\xi)$, so the integration constant appears to be zero).

Using known analytical properties of Cauchy integral[22] near the edge of a contour and the symmetry of $b(\xi)$, one may see from Eq. (11) that

$$b(\xi) - b(0) = -\frac{1}{\pi}\left(\frac{1}{q+p}\right)^{1/p} b^{(1-q)/p}(0)\cot\frac{\pi}{2p}\left|\xi\right|^{1/p} \tag{12}$$

when $\xi \to 0$ (here $p > 1/2$). If the function $b(\xi)$ has tails at infinities, then one obtains immediately from Eq. (11)

$$b(\xi) = (q+p)^{1/(1-q)}\left[\int_{-\infty}^{\infty} d\zeta b(\zeta)\right]^{p/(1-q)} |\xi|^{-(p+1)/(1-q)} \tag{13}$$

at $|\xi| \to \infty$.

If $q < 1$ (including the physically interesting case $q = 0$), there are the feature (12) at the center and the tails (13) at $|\xi| \to \infty$. (Note that $\int_{-\infty}^{\infty} d\zeta b(\zeta) < \infty$ since $(p+1)/(1-q) > 1$ for $q < 1$.) In particular, when $q = 0, p = 1$, (1) and (11) turns to be linear equations, and (11) has a simple solution

$$b(\xi) = \frac{const}{\xi^2 + \pi^2}. \tag{14}$$

The relation (13) is obviously impossible if $q \geq 1$, and in this case $b(\xi)$ may be nonzero only near the point $\xi = 0$, so $b(\xi) = 0$ if $|\xi| \geq \xi_0 > 1$. When $q \geq 1$, we shall search the solution among such functions. It is convenient to use the function $h(\xi)$ which is nonzero in the range $-1 < \xi < 1$, instead of $b(\xi)$:

$$b(\xi) = \xi_0^{1/(q+p-1)} h(\xi/\xi_0). \tag{15}$$

The parameter ξ_0 will be found at the end of our calculations. The integral in Eq. (11) becomes the integral between the limits -1 and 1. Using known analytical properties of the Cauchy integral near the edges of the contour of integration, we see from Eq.(11) that $h(\xi) \propto \sqrt{1-\xi^2}$ near the points $\xi = \pm 1$ when $q = 1$. $h(\xi \sim \pm 1)$ can also be found in the case $q > 1$. In particular,

$$h(\xi) = \left(\frac{1}{q+p}\right)^{1/(p+q-1)} \left(\frac{q-1}{q+p-1}\right)^{p/(p+q-1)} \times \left(\ln \frac{1}{1-\xi}\right)^{-p/(p+q-1)} \tag{16}$$

when $1 - \xi \ll 1$. We do not discuss this case in detail.

Let us consider the most interesting case $q = 1$, for which it appears possible to find an exact solution[18]. One may invert the Gilbert transformation, using relations for functions restricted at the edges of a finite interval[22]. Then we obtain

$$h(\xi) = \frac{1}{\pi^2}\left(\frac{1}{p+1}\right)^{1/p}\sqrt{1-\xi^2} \times \int_{-1}^{1} d\zeta \frac{1}{\sqrt{1-\zeta^2}} \frac{|\zeta|^{1/p}\, sign(\zeta)}{\zeta - \xi} = \frac{1}{\pi^2 p}\left(\frac{1}{p+1}\right)^{1/p} \times B\left(\frac{1}{2}, \frac{1}{2p}\right)\sqrt{1-\xi^2}\, {}_2F_1\left(\frac{1}{2} - \frac{1}{2p}, 1; \frac{3}{2}; 1-\xi^2\right), \tag{17}$$

where $B(,)$ is the beta-function and ${}_2F_1$ is the hypergeometric function. $h(0) = \pi^{-2}(p+1)^{-1/p} B\left(\frac{1}{2}, \frac{1}{2p}\right)$. Now we recall that $\Phi = \int dx B(x,t) = const$. Thus,

$$\xi_0 = \pi^{p/(p+1)} (p+1) B^{-p/(p+1)} \left(\frac{1}{2}, \frac{1}{2p}\right). \tag{18}$$

The relations (9), (15), (17), and (18) give an exact solution of the problem under consideration in the case $q = 1$.

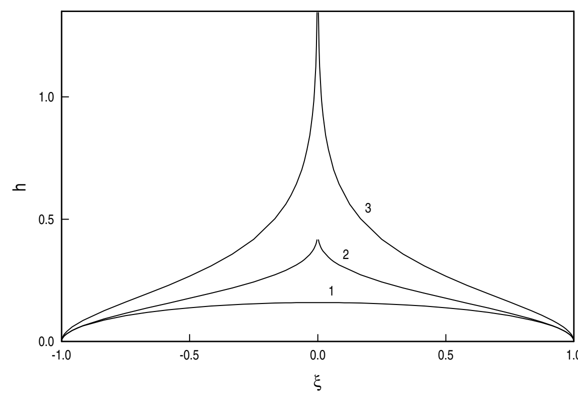

FIG. 1. Scaling functions $h(\xi)$ determining the evolution of the magnetic flux injected into a type-II superconductor film (see Eqs. (9), (15), (17), and (18)). Curves $1, 2, 3$ correspond to: $q = p = 1$ – the case of a soft superconductor; $q = 1, p = 3$; and $0 \leq q < \infty, p \to \infty$ – the case of a hard superconductor.

Figs. 1 and 2 show scaling solutions $h(\xi)$ for $q = 1$ and different p. The case $p \geq 1$ (Fig. 1) is interesting for us. The solutions $h(\xi)$ with two maxima for $0 < p < 1$ (Fig. 2) are shown here for generality. In particular, in the case of a soft superconductor ($q = p = 1$)

$$h(\xi) = \frac{\sqrt{1-\xi^2}}{2\pi}, \quad \xi_0 = 2. \tag{19}$$

When $q = 1, p \to \infty$

$$h(\xi) = \frac{2}{\pi^2} \ln\left|\frac{1+\sqrt{1-\xi^2}}{\xi}\right|, \quad \xi_0 = \pi/2, \tag{20}$$

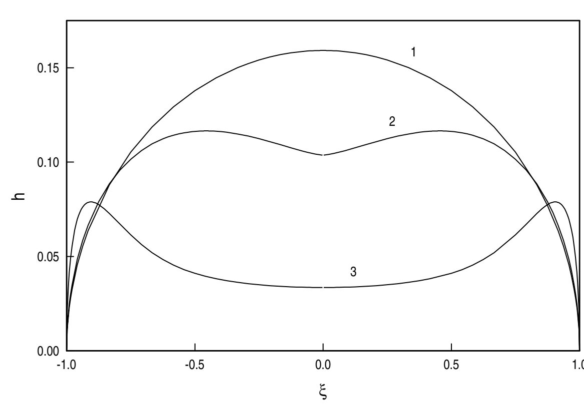

FIG. 2. Same as Fig. 1 but curves $1, 2, 3$ correspond to: $q = p = 1$; $q = 1, p = 0.6$; and $q = 1, p = 0.11$.

and we obtain the solution for the case of the hard superconductor at low temperatures. This limit has the same form also for other values $0 \leq q < \infty$. Here, in each

point of the distribution, the integral $\int d\zeta h(\zeta)/(\zeta-\xi) = const$, so we are concerned with a nonlocal analog of the sandpile problem. As we have noted above, at $q=1$, the solution has a square-root form near the fronts. Now we may see from Eq. (17) that

$$h(\xi \sim \pm 1) \cong \frac{1}{\pi^2 p}\,(p+1)^{-1/p}\,B\left(\frac{1}{2},\frac{1}{2p}\right)\sqrt{1-\xi^2}. \quad (21)$$

Now let us show how the current density distribution evolves. When $q \geq 1$, one may find from the relations (6), (9), and (15) that

$$j(x,t) = \frac{c}{2\pi^2 d}\left(\frac{tD}{\Phi}\right)^{1/(q+p)} \times \xi_0^{1/(q+p-1)} g\left(\frac{x}{\xi_0\left(\Phi^{q+p-1}tD\right)^{1/(q+p)}}\right), \quad (22)$$

where $g(\xi) \equiv -\int_{-1}^{1} d\zeta h(\zeta)/(\zeta-\xi)$.; When $|\xi| \leq 1$ and $q=1$, $g(\xi) = (p+1)^{-1/p}\,|\xi|^{1/p}\,sign(\xi)$. If $q=1$ and $|\xi| \geq 1$, using (17) one gets

$$g(\xi) = \frac{1}{\pi}(p+1)^{-(p+1)/p}B\left(\frac{1}{2},\frac{1}{2p}\right) \times \frac{1}{\xi}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}+\frac{1}{2p};\frac{3}{2}+\frac{1}{2p};\frac{1}{\xi^2}\right) \quad (23)$$

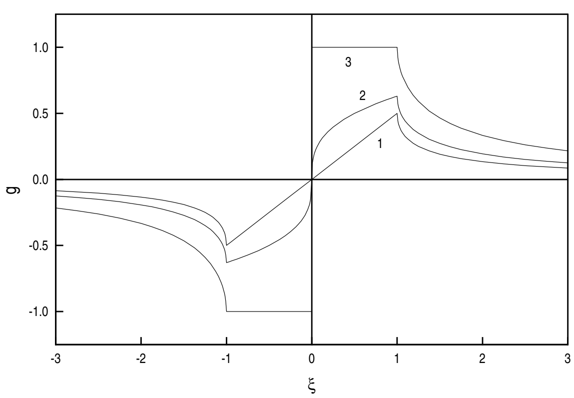

FIG. 3. Scaling functions $g(\xi)$ (see Eq. (23) for the current density distributions corresponding to the flux distributions shown in Fig. 1. Curves $1,2,3$ correspond to: $q=p=1$; $q=1, p=3$; and $0 \leq q < \infty, p \to \infty$.

(see Fig. 3). Near the points $\xi = \pm 1$ (at $|\xi| \geq 1$),

$$g(\xi) - g(|\xi|=1) \cong -\frac{1}{\pi p}(p+1)^{-1/p}B\left(\frac{1}{2},\frac{1}{2p}\right)\sqrt{|\xi|-1}\,sign(\xi). \quad (24)$$

In particular, if $q=p=1$, then $g(\xi) = \xi/2$ for $|\xi| \leq 1$ and $g(\xi) = \left[2\xi\left(1+\sqrt{1-\xi^{-2}}\right)\right]^{-1}$ for $|\xi| \geq 1$. In the limit $p \to \infty, q < \infty$, $g(\xi) = sign(\xi)$ for $|\xi| \leq 1$ and $g(\xi) = (2/\pi)\arcsin(1/\xi)$ for $|\xi| \geq 1$. In the linear case $q=0, p=1$, as can be seen from Eq. (14), the current density (6) is determined by the integral $-\int_{-\infty}^{\infty} d\zeta b(\zeta)/(\zeta-\xi) = const\,\xi/\left(\xi^2+\pi^2\right)$.

Remarkably, in the studied problem, nonspecified initial distributions of the magnetic flux evolve to unified space-time distributions (9) at long times. It is significant here that $\Phi \neq 0$. If total flux of the initial distribution is zero, then the distribution evolves to a different scaling form at long times.

The duration of the transition from the initial distribution of the injected magnetic flux to the scaling solutions (see Fig. 1 and Fig. 2) depends on the specific form of the initial distribution. Obviously, in the case $q \geq 1$, the initial distributions should be localized to obtain scaling solutions at long times (see the discussion after Eq. (14).

In conclusion, let us compare the obtained solutions with the solutions of the local equation

$$\frac{\partial B}{\partial t} = D\frac{\partial}{\partial x}\left[|B|^q\,|\partial B/\partial x|^p\,sign\left(\frac{\partial B}{\partial x}\right)\right] \quad (25)$$

describing evolution of the magnetic flux injected into a type-II superconductor plate when the flux lines are parallel to the surfaces. (We assume that the problem is homogeneous along a surface.) Let the flux be injected insite of a plate. In this case, until the vortices reach the boundaries, $B(x,t) = C_1 t^{-1/(q+2p)} b\left(x/C_2 t^{1/(q+2p)}\right)$ where C_1 and C_2 are constants, and one can see that $b(\xi) \propto \left[1-(\xi/\xi_0)^{(1+p)/p}\right]^{p/(q+p-1)}$. This function has no such striking peculiarities at the center as in Figs. 1 and 2.

We consider above the homogeneous along y axis problem. One can also obtain the exponents α and β introduced in Eq. (7) for the situation, in which the initial distribution of the injected flux is localized not along a line but in the vicinity of some point on a film, so instead of the coordinate x now one can introduce the radius r. Then, as it can be shown, in the case of nonlinear nonlocal diffusion in a film, $\alpha = \beta/2 = 1/(2q+2p-1)$ (it should be $q+p > 1/2$). For a parallel geometry (i.e. for nonlinear local diffusion in a plate) one can also search for scaling solutions with axial symmetry. In this problem, $\alpha = \beta/2 = 1/(2q+3p-1)$, $2q+3p > 1$.

As it can be easy understood, the problem described by Eq. (1) is a direct generalization of the evolution problem for charge distribution placed on a conducting plane with dissipative transport. Analogical problems for charges injected into a wire and for evolution of yz homogeneous charge distributions in 3D can be also formulated. The equation for the first one is of the form

$$\frac{\partial n(x)}{\partial t} = D\frac{\partial}{\partial x}\left\{n(x)\int_{-\infty}^{\infty} du\frac{n(u)}{|u-x|(u-x)}\right\}, \quad (26)$$

and the last one is described by the following equation,

$$\frac{\partial n\,(x)}{\partial t}=D\frac{\partial}{\partial x}\left\{n(x)\int_{-\infty}^{\infty}du\,n\,(u)\,sign(u-x)\right\}, \quad (27)$$

where $n(x)$ is the charge distribution. The scaling solutions of Eqs (26) and (27) are of the form (9) with $\alpha=\beta=1/2$. The scaling function $b(\xi)$ looks like

$$b(\xi)\propto(\theta(\xi+\xi_0)-\theta(\xi-\xi_0)) \quad (28)$$

for the solution of Eq. (26), where $\theta(\xi)$ is a step function, and

$$b(\xi)\propto(\delta(\xi+\xi_0)+\delta(\xi-\xi_0)) \quad (29)$$

for the solution of Eq. (27). One may compare this functions with answers for Eq. (1) shown in Figs 1 and 2. Note that an exact solution can be found generally for the equation of the form: $\partial B/\partial t=\partial\left(|B|\,F\{B\}\right)/\partial x$, if the condition $F\{cB\}=c^pF\{B\}$ is satisfied ($c=const\neq 0, p=const>0$).

In summary, we have considered problem of nonlinear nonlocal diffusion of the magnetic flux injected in an infinite thin type-II superconductor. We have solved it exactly in the most interesting case of flow resistivity proportional to the local magnetic induction B. The obtained flux space-time distributions are of the self-similar form with rather striking scaling functions. Two questions remain open: how can the obtained distributions be observed experimentally, and how do the edges of a thin strip affect our solutions?

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